# Stability analysis in the Perturbed Robe's Finite Straight Segment model under the effect of Viscosity


by

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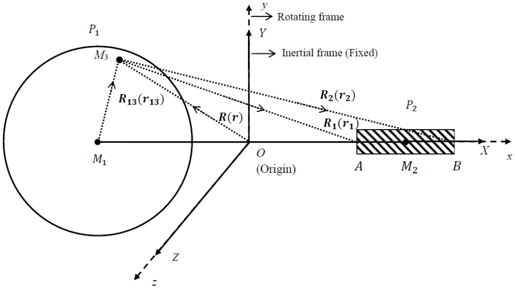


Figure 1: Configuration

Forces acting on $P_3$:

Buoyancy Force: $\mathbf{F}_B = \dfrac{4\pi\rho_1^2 G m_3^*}{3\rho_3}\mathbf{R}_{13}$,

Attraction of $P_2$: $\mathbf{F}_{32} = \dfrac{2Gm_2^* m_3^*}{(R_1+R_2)^2 - 4l^2}\left(\hat{\mathbf{R}}_1 + \hat{\mathbf{R}}_2\right)$,

Attraction of fluid: $\mathbf{F}_A = -\dfrac{4\pi\rho_1 G m_3^*}{3}\mathbf{R}_{13}$.

Considering all above forces, Viscosity, and perturbation in the Coriolis and centrifugal forces, the equations of motion in rotating coordinate system $Oxyz$ with dimensionless variables are:

$$\left.\begin{aligned} \ddot{x} - 2n\phi_1\dot{y} &= -\alpha\dot{x} + W_x, \\ \ddot{y} + 2n\phi_1\dot{x} &= -\alpha\dot{y} + W_y, \\ \ddot{z} &= -\alpha\dot{z} + W_z, \end{aligned}\right\} \qquad (1)$$

where

$$W = \begin{cases} {}_lW & \text{when } l \neq 0, \\ {}_0W = \lim_{l\to 0}({}_lW) & \text{when } l = 0, \end{cases}$$

$$\begin{aligned} {}_lW =& \frac{1}{2}n^2\phi(x^2+y^2) - \frac{k}{2}[(x+\mu)^2+y^2+z^2] + \\ & \frac{\mu}{2l}\log\left(\frac{r_1+r_2+2l}{r_1+r_2-2l}\right), \\ {}_0W =& \frac{1}{2}n^2\phi(x^2+y^2) - \frac{k}{2}[(x+\mu)^2+y^2+z^2] + \\ & \frac{\mu}{\sqrt{(x-1+\mu)^2+y^2+z^2}}, \\ k =& \frac{4\pi}{3}\rho_1\left(1-\frac{\rho_1}{\rho_3}\right), \text{ density parameter} \\ n =& 1+\frac{l^2}{2},\ l << 1,\ \mu = m_2^*,\ \ 0<\mu<1, \\ r_1^2 =& (x-1+\mu+l)^2+y^2+z^2, \\ r_2^2 =& (x-1+\mu-l)^2+y^2+z^2, \\ \phi_1 =& 1+\epsilon_1,\ |\ \epsilon_1\ |\ll 1 \text{ and } \phi = 1+\epsilon,\ |\ \epsilon\ |\ll 1. \end{aligned}$$

Here, $\phi_1$ and $\phi$ are the perturbing parameters in the Coriolis and centrifugal forces respectively and $\alpha$, a positive constant, is the viscosity parameter.

**Two Collinear equilibrium points**: $L_{r_1}(x_1, 0, 0)$ and $L_{r_2}(x_2, 0, 0)$,

$$x_1 = -\mu + \frac{\mu(1+l^2)}{1-k+2\mu+l^2(1+4\mu)}\epsilon, \tag{2}$$

$$x_2 = \frac{1}{2(k-1-l^2)}\left[(1+l^2)(\mu-2)+2k(1-\mu)-\sqrt{B+2l^2[2+2k^2+2k(\mu-2)-4\mu+\mu^2]}\right]+q\epsilon, \tag{3}$$

with $B = \mu(4k+\mu-4)$ and $q$ is given by Eq. (4).

$$q = \frac{-l^2\left(\frac{-2k^3-2k^2\mu+6k^2-k\mu^2+2k\mu-k\mu\sqrt{B}-6k+2}{2(k-1)^2\sqrt{B}}+\frac{-2k\mu-\sqrt{B}+2k+\mu-2}{2(k-1)}\right)-\frac{\left(-2k\mu-\sqrt{B}+2k+\mu-2\right)}{2(k-1)}}{l^2\left(1-\frac{f}{\sqrt{B}\left(\sqrt{B}+\mu\right)^5}\right)+\frac{16(k-1)^3\mu}{\left(\sqrt{B}+\mu\right)^3}-k+1}, \tag{4}$$

$$\begin{aligned} f =& 32(k-1)^2\mu\big(3k^3\mu-k^3\sqrt{B}+9k^2\mu^2-9k^2\mu+3k^2\mu\sqrt{B}+3k^2\sqrt{B}+3k\mu^3-9k\mu^2+3k\mu^2\sqrt{B} \\ &+9k\mu-3k\mu\sqrt{B}-3k\sqrt{B}+\sqrt{B}-3\mu\big). \end{aligned}$$

The point $L_{r_1}(x_1, 0, 0)$ is always an equilibrium point, whereas $L_{r_2}(x_2, 0, 0)$ will be an equilibrium point provided $k > 1 + l^2$.

**Infinite Non-Collinear equilibrium points**: $L_{r_3}(x, y, 0)$ $(y \neq 0)$, which lie within the spherical shell as well as on the circle given by following equation

$$(1-\mu-x)^2 + y^2 = \left[\left(1-\frac{1}{3}\epsilon\right)\left(1-\frac{1}{3}l^2\right)\right]^2 \tag{5}$$

provided $k = (1+\epsilon)(1-\mu)(1+l^2)$.
To sum up, a non-collinear equilibrium point has coordinates $L_{r_3}(a, b, 0)$, where

$$a = 1-\mu-\left(1-\frac{\epsilon}{3}\right)\left(1-\frac{1}{3}l^2\right)\cos\theta,\ \ b = \left(1-\frac{\epsilon}{3}\right)\left(1-\frac{1}{3}l^2\right)\sin\theta.$$

**Two Out-of-plane equilibrium points**: $L_{r_4, r_5}\left(\frac{k}{\phi}(1-l^2), 0, \pm\sqrt{b_1^2 - a_1^2}\right)$,
$a_1 = 1-\mu+k(\epsilon-1)$ and $b_1 = \left(-\frac{\mu}{k} + \frac{\mu l^2}{k-\phi(1-\mu)}\right)^{1/3}$
exist provided $k < 0$ and $k + \mu + 2\mu l^2 > 0$.

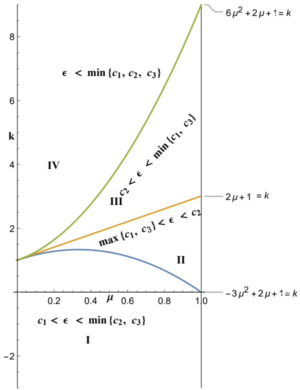


Figure 2: Bifurcation diagram for Theorem 1 showing the stability conditions for the equilibrium position $L_{r_1}$ in $\mu k$-plane.

## Theorem 1 (Stability criterion for $L_{r_1}$)

*Let $k \neq 1+2\mu$ and $c_2 \neq c_3$.*

1. *For $k < 1+2\mu-3\mu^2$, the equilibrium position $L_{r_1}(x_1,0,0)$ of the system (1) is asymptotically stable if $c_1 < \epsilon < min\{c_2,c_3\}$ and stable if $\epsilon = c_1$ or $\epsilon = min\{c_2,c_3\}$.*
2. *For $1+2\mu-3\mu^2 < k < 1+2\mu$, the equilibrium position $L_{r_1}(x_1,0,0)$ for the system (1) is asymptotically stable if $max\{c_1,c_3\} < \epsilon < c_2$ and stable if $\epsilon = c_2$ or $\epsilon = max\{c_1,c_3\}$.*
3. *For $1+2\mu < k < 1+2\mu+6\mu^2$, the equilibrium position $L_{r_1}(x_1,0,0)$ for the system (1) is asymptotically stable if $c_2 < \epsilon < min\{c_1,c_3\}$ and stable if $\epsilon = c_2$ or $\epsilon = min\{c_1,c_3\}$.*
4. *For $k > 1+2\mu+6\mu^2$, the equilibrium position $L_{r_1}(x_1,0,0)$ for the system (1) is asymptotically stable if $\epsilon < min\{c_1,c_2,c_3\}$ and stable if $\epsilon = min\{c_1,c_2,c_3\}$.*

Where

$$\left.\begin{aligned}
c_1 &= -\frac{\mu + k + 2\mu l^2}{3\mu A_0}, \\
c_2 &= -\frac{l^2(1 + 4\mu) + 1 + 2\mu - k}{1 + 6\mu A_0}, \\
c_3 &= -\frac{l^2(1 - 2\mu) + 1 - \mu - k}{1 - 3\mu A_0}, \\
A_0 &= \frac{\mu}{1 + 2\mu - k}.
\end{aligned}\right\} \tag{6}$$

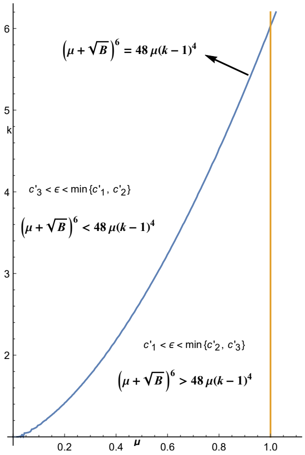


Figure 3: Bifurcation diagram for Theorem 2 showing the stability conditions for the equilibrium position $L_{r_2}$ in $\mu k$-plane.

**Theorem 2 (Stability criterion for $L_{r_2}$)**

*Let $k > 1 + l^2$ and $c'_2 \neq c'_3$.*

1. *For $(\mu + \sqrt{B})^6 < 48(k-1)^4\mu$, the equilibrium position $L_{r_2}(x_2, 0, 0)$ of the system (1) is asymptotically stable if $c'_3 < \epsilon < min\{c'_1, c'_2\}$ and stable if $\epsilon = c'_3$ or $\epsilon = min\{c'_1, c'_2\}$.*
2. *For $(\mu + \sqrt{B})^6 > 48(k-1)^4\mu$, the equilibrium position $L_{r_2}(x_2, 0, 0)$ of the system (1) is asymptotically stable if $c'_1 < \epsilon < min\{c'_2, c'_3\}$ and stable if $\epsilon = c'_1$ or $\epsilon = min\{c'_2, c'_3\}$.*

Where

$$c_1' = \frac{A_2 - 2A_1 - k}{A_4},\ c_2' = \frac{k - 4A_1 + 2A_2 - 1 - l^2}{A_3},\ c_3' = \frac{k + 2A_1 - A_2 - 1 - l^2}{A_4}, \tag{7}$$

with

$$A_1 = \frac{4\mu(k-1)^3}{(\mu+\sqrt{B})^3},\ A_3 = 1 + \frac{96\mu(k-1)^4}{(\mu+\sqrt{B})^6},\ A_4 = 1 - \frac{48\mu(k-1)^4}{(\mu+\sqrt{B})^6},$$

$$A_2 = \frac{1}{\sqrt{B}\left(\sqrt{B}+\mu\right)^5}\left[16(k-1)^2 l^2 \mu \left\{3(k-1)\mu\left(k^2 + k\left(\sqrt{B}-2\right)+1\right) + 3k\mu^3 + 3k\mu^2\left(\sqrt{B}+3k-3\right) + (k-1)^3\left(-\sqrt{B}\right)\right\}\right].$$

Non-collinear equilibrium points : Unstable

Out-of-plane equilibrium points : Unstable

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